The Role of Pricing in Differentiated Services Networks

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Abstract

We study the role of pricing in packet-based telecommunication networks which use priorities to provide differentiated Quality-of-Service (QoS). Customers are given the freedom to choose priorities, but are charged accordingly. Using a game theoretic framework, we study the case where customers choose an allocation of priorities to packets in order to optimize their net benefit. For the single link case, we show that there always exists a (Nash) equilibrium for the corresponding non-cooperative game, which, however, is not necessarily unique. Furthermore, we show that under each equilibrium, packet loss depends only on pricing decisions and therefore pricing becomes a tool to control link congestion.

Keywords: Differentiated Services, Quality-of-Service, Congestion Control, Pricing.

1 Introduction

In this paper, we study the role of pricing in Differentiated Services (DiffServ) networks. DiffServ has been proposed by the Internet Engineering Task Force (IETF) as a mechanism to provide Quality-of-Service (QoS) in packet-based networks (such as the Internet). The basic idea behind DiffServ is the following. Networks deploying DiffServ offer a (small) number of different service classes. Users accessing the network assign (mark) their traffic to one of these classes, and the network forwards packets at each node according to class-specific rules (called per-hop behaviors) [9]. This mechanism can be used to provide differentiated QoS. For example, DiffServ could be used to implement a premium and a best-effort service, where premium traffic always gets preferential treatment over best-effort traffic [12]. As a result, premium traffic will obtain a better QoS when congestion in the network increases, i.e. premium traffic experiences less delay and is less likely to be dropped. A range of other service classes are also possible and have been proposed [6, 7, 8]. An important feature of DiffServ is that by aggregating a multitude of flows into a small number of services classes, DiffServ eliminates the need to recognize and store information about each individual flow in core routers; and makes DiffServ scalable.

In the DiffServ framework, it is natural to assume that users are charged based on the service classes that they use, i.e. a high-quality (premium) service is (significantly) more expensive than a low-quality (best-effort) service. Also, one would expect that users choose service classes according to their service requirements, i.e. users try to meet their QoS requirements at the lowest possible cost. In this paper we analyze this situation. In particular, we are interested in the following questions,

(a) Does the resulting interaction among users lead to a stable operation of the network?

(b) How do pricing decisions influence the performance of the network, e.g. the QoS in the different service classes?

These issues are important for the understanding of DiffServ; however, they haven't been adequately addressed from a theoretical point of view. This paper is an attempt to make a step into this direction.

For our analysis, we model DiffServ as a priority service, and consider a network which offers a finite set $\mathcal{I} = \{1, ..., N\}$ of priority levels. At each node, packets are served according to a priority rule, so that packets with high priorities will experience less delay and are less likely to be dropped. Users are free to choose the priorities for their traffic, but are charged accordingly by the network. In particular, the network charges a price u_i per unit traffic submitted with priority i, where for $i, j \in \mathcal{I}$ we have

$$u_i < u_j, \qquad \text{when } i < j.$$

An important feature of this model is that the prices u_i, $i \in \mathcal{I}$, are static and do not change (on a fast time-scale) to respond to changes in the network load. However users can change their allocation dynamically (on a fast time-scale) to maximize their net benefit from using the network.

To study this priority service, we adopt an economic framework where we characterize the users' service requirements by utility functions. Roughly, submitting more high priority packets will increase the utility of an individual user. However, doing so will also increase the cost the user has to pay to the network provider. It is then natural to assume that users will choose an allocation of packets to priorities to maximize their net benefit, i.e. their utility minus cost. We model this situation as a non-cooperative game for which we wish to determine whether there exists an equilibrium, and how pricing decision influence the characteristic of an equilibrium. To keep the analysis simple, we consider only the single link case. The main contributions of the paper are the following.

(1) We develop a mathematical model for studying DiffServ in a market-based environment where users are charged depending on the service class that they use.

(2) We show that pricing decision do have an influence on the network performance. In particular, our analysis reveals a surprisingly simple relation between the prices of the different priority levels and the packet loss probabilities.

The framework that we consider here is closely related to the Paris Metro Pricing (PMP) proposal [13, 14] for providing QoS in the Internet. PMP partitions the network into several logically separated channels where channels differ in the prices paid for using them. This can be implemented by using the priority scheme, combined with pricing, described above. However, [13, 14] do not analyze the existence of an equilibrium and the role of pricing. [2] provides simulation results to demonstrate that priorities can be an efficient tool for providing QoS in packet-based networks. In particular, [2] illustrates that using priorities, combined with pricing, achieves a higher a social welfare than a best-effort/flat-rate scheme. Gupta et. al. investigate in [4] a priority service similar to the one propose here, however they consider the situation where the network dynamically changes the prices associated with the different priority classes in order to track a socially optimal allocation. This approach is theoretically appealing, but expensive to implement. The scheme considered in this paper is simpler as prices associated with the different service classes are fixed. Park et. al. also consider in [1, 15] a service where packets get marked according to customer's QoS requirements. However, there is no price associated with the different traffic classes, i.e. the costs incurred to customers are purely performance related. They study the situation where users are free to choose their traffic class [15], as well as where users indicate their QoS requirements and a network controller assigns network resources [1].

The priority service that we consider here can be interpreted as an approximation to the smart market proposal of MacKie-Mason and Varian [10, 11]. While in the smart market users can attach any bid $u \in \Re_+$ to their packets, in the above priority service users are limited to choose a bid $u \in \{u_1, ..., u_N\}$. Furthermore, while in the smart market users pay only for packets that get transmitted, in the above priority service users are charged for all submitted packets. Finally, while in the smart market users are charged a price that is equal to the highest bid among the dropped packets, here users are charged the bid price that they attached to their packets. Despite these differences, which make the implementation of the priority service much easier, we will show that users (with elastic traffic) pay the same price per transmitted packet, and achieve the same average throughput, under the two service mechanisms.

The rest of the paper is organized as follows. In Section 2 we present the link model that we use for our analysis. In Section 3 we study the case where all users accessing the link measure quality of service solely as a function of the average throughput they obtain. In Section 4 we extend this analysis to the case where

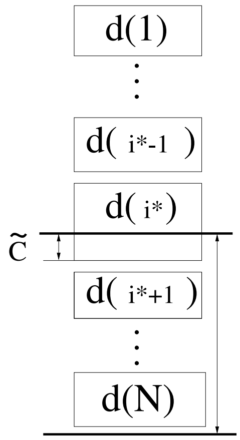

Figure 1: The link transmits all packets with a priority higher than i^*. From the packets with priority i^*, only $\tilde{C} = \left(C - \sum_{i=i^*+1}^{N} d(i)\right)$ packets are transmitted, which are chosen at random. All other packets are dropped.

users also care about packet loss as a measure for quality of service. In Section 5 we present the conclusions. Proofs for the results in Section 3 and 4 are given in Appendix A-C.

2 Link Model

We use a discrete-time model for our analysis, where time is divided into slots. Let $C > 0$ be a given constant, and consider a single link with a fixed capacity of C packets per time slot, where packets can be arbitrarily small. There is no buffering and packets which do not get transmitted in a given time slot are dropped. The link supports a finite set $\mathcal{I} = \{1, ..., N\}$ of priorities, where 1 is the lowest priority - 2 the second lowest, and so on. Assume that R users share the link, and assume that each user submits packets at a constant and deterministic rate. In particular, let $d_r = (d_r(1), ..., d_R(N)) \in \Re_+^N$, $r = 1, ..., R$, be the allocation of user r, where $d_r(i)$ is the number of packets user r submits in each time slot in priority i, and let $(d_1, ..., d_R)$ be the corresponding allocation vector. Furthermore, let $d = (d(1), ..., d(N))$ be the aggregated allocation (over all users), where

$$d(i) = \sum_{r=1}^{R} d_r(i), \qquad i = 1, ..., N,$$

is the total number of packets submitted in priority i in each time slot. The link then serves packets as follows (see Figure 1 for an illustration). In each time slot, all packets with priority i such that

$$\sum_{k=i}^{N} d(k) < C$$

are transmitted. For priority i^* with

$$\sum_{i=i^*+1}^{N} d(i) < C \quad \text{and} \quad \sum_{i=i^*}^{N} d(i) \geq C,$$

only $\left(C - \sum_{i=i^*+1}^{N} d(i)\right)$ packets, which are chosen at random, get transmitted. All packets with priority $i < i^*$ are dropped. For the aggregated allocation $d \in \Re_+^N$, the (packet) transmission probability $P_{tr}(i, d)$ in priority i is then given as follows,

$$P_{tr}(i,d) = \begin{cases} 1 & \text{if } C > \sum_{k=i}^{N} d(k), \\ \frac{C - \sum_{k=i+1}^{N} d(k)}{d(i)} & \text{if } \sum_{k=i}^{N} d(k) \geq C > \sum_{k=i+1}^{N} d(k), \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_{tr}(d) = (P_{tr}(1,d), ..., P_{tr}(N,d))$ be the corresponding probability vector. The average throughput of user r is then given by

$$x_r = \sum_{i=1}^{N} d_r(i) P_{tr}(i,d), \qquad r = 1, ..., R.$$

Users are charged a price u_i per submitted packet with priority i, where for $i, k \in \mathcal{I}$, $k < i$, we have $0 < u_k < u_i$. In addition, there exists a feedback mechanism that allows users to detect whether their packets were transmitted and to estimate the packet transmission probabilities in each priority.

The above model describes a priority service which charges a fixed price u_i is per submitted packet with priority i. However the price for actually transmitting a packet will depend on the total demand: in order to transmit a packet in priority i, users have to submit a packet (on average) $1/P_{tr}(i,d)$ times and the (average) price for transmitting a packet in priority i is then equal to $u_i/P_{tr}(i,d)$. In the following, we study this service.

3 Customers with Elastic Traffic

We first analyze the above service model for users with elastic traffic [16], i.e. for users who perceive quality of service solely as a function of the average throughput. For this case, we can characterize the service preferences of each user through a quality indicator (utility function) which depends on a single variable: the average throughput. We will assume that these utility functions are increasing in the average throughput. Examples of users with elastic traffic are users sending email, transferring files, and browsing the Web.

Consider a fixed user $r \in \{1, ..., R\}$. Furthermore, suppose that $d_r = (d_r(1), ..., d_r(N)) \in \Re_+^N$ is the allocation of user r and $d = (d(1), ..., d(N)) \in \Re_+^N$ is the aggregated allocation. Then, we associate with user r the utility function $U_r(x_r)$, where

$$x_r = \sum_{i=1}^{N} P_{tr}(i,d) d_r(i)$$

is the average throughput of user r. We make the following technical assumption which ensures that utility functions of all users are increasing and concave in the average throughput.

Assumption 1 *For each user $r = 1, ..., R$, the function $U_r : \Re_+ \mapsto \Re_+$, is given by*

$$U_r(x_r) = \int_0^{x_r} G_r(x) dx,$$

where $G_r(x) : \Re_+ \mapsto \Re_+$ is bounded, strictly decreasing and differentiable with

$$G_r(0) > 0 \quad \text{and} \quad \lim_{x \to \infty} G_r(x) = 0.$$

Note that Assumption 1 does not require that all users have the same utility function.

Assumption 1 implies that a user can achieve a high utility by submitting high priority packets; however, doing so will also increase the cost. Users therefore face a trade-off between achieving a high utility and keeping their cost low. We capture this situation as follows. Given an aggregated demand $d = (d(1), ..., d(N)) \in \Re_+^N$, and an allocation $d_r = (d_r(1), ..., d_r(N)) \in \Re_+^N$ of user r, we associate with user r a net benefit function given by

$$U_r(x_r) - \sum_i d_r(i) u_i,$$

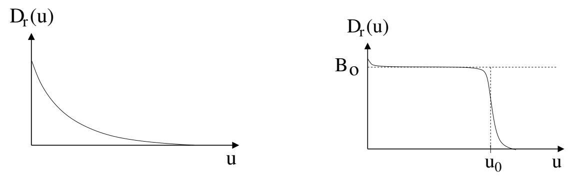

Figure 2: Demand function for users with data traffic (a), and users with voice traffic (b). The demand of users with data traffic decreases as the price u increases. As voice requires a constant transmission rate, denoted as B_0 here, the demand $D_r(u)$ of users with voice traffic will (essentially) be equal to B_0 as long as the price u does not exceed the threshold u_0 (the maximal price the user is willing to pay for the voice connection), and then drop to 0.

where

$$x_r = \sum_{i=1}^{N} P_{tr}(i,d) d_r(i).$$

In the following, we assume that each user chooses an allocation that maximizes its own net benefit. Note that the net benefit functions of the individual users are couple through the transmission probabilities $P_{tr}(i,\cdot)$. Therefore, changing d_r will not only affect the net benefit of user r, but the net benefit functions of all users. In the next subsection, we study this interaction among users.

We introduce some more notation. For $r = 1, ..., R$, let the function $D_r : \Re_+ \rightarrow \Re_+$ be such that $D_r(u)$ is the optimal solution the the maximization problem

$$\max_{x \geq 0} \{U_r(x) - xu\}, \quad u \in \Re_+.$$

$D_r(u)$ can be interpreted as the demand function of user r, i.e. $D_r(u)$ is the number of packets users r submits when the price per packet is u and all packets are transmitted with probability 1. Figure 2 gives two demand functions that are possible under Assumption 1. Note that the function $G_r(x)$ in Assumption 1 is the inverse of the function $D_r(u)$. Finally, we use $D(u)$, $u \in \Re_+$, given by

$$D(u) = \sum_{r=1}^{R} D_r(u),$$

to denote the aggregated demand over all users under the price u.

3.1 Formulation as a Non-Cooperative Game

In this subsection, we study the case where all users simultaneously try to maximize their own net benefit. We model this situation as a non-cooperative game. For an introduction to the basic concepts of game theory we refer to [3]. For the analysis, we assume that customers ignore how they influence the transmission probabilities when optimizing their own net benefit. This simplifying assumption corresponds to the standard competitive price-taking assumption of economic theory. In this context, it can be justified when

(a) many "small" users access the link so that the influence of an individual user on the transmission probabilities is negligible.

(b) it is impossible (or too expensive) for a customer to determine how the choice of an allocation may affect the transmission probabilities.

We then define an equilibrium allocation as follows.

Definition 1 *Let $(d_1^*, ..., d_R^*)$ be a given allocation vector such that $d^*(r) \in \Re_+^N$, $r = 1, ..., R$, and let $d^* = (d^*(1), ..., d^*(N))$,*

$$d^*(i) = \sum_{r=1}^{R} d_r^*(i), \qquad i = 1, ..., N,$$

be the corresponding aggregated allocation. We call $(d_1^*, ..., d_R^*)$ an equilibrium allocation when for every user $r = 1, ..., R$, d_r^* is an optimal solution to the maximization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^{N} d_r(i) u_i \right\},$$

where

$$x_r = \sum_{r=1}^{R} d_r(i) P_{tr}(i, d^*).$$

Under an equilibrium allocation $(d_1^*, ..., d_R^*)$, the allocation d_r^*, $r = 1, ..., R$, maximizes the net benefit of user r. Therefore, no user has an incentive to change its allocation d_r^* as this would decrease the net benefit. This implies that once an equilibrium is reached, no user will deviate.

We make the following assumption to exclude the trivial case where the link capacity exceeds the the demand in the lowest priority level.

Assumption 2 *We have $D(u_1) > C$.*

In the next proposition, we will derive the necessary conditions for an allocation vector to be an equilibrium allocation. This will allow us to characterize an equilbiritum allocation, and determine how pricing decision influence the properties of an equilibrium.

Proposition 1 *Let Assumption 1 and 2 hold. In particular, let*

$$u_{r,\max} = G_r(0), \qquad r = 1, ..., R.$$

Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let

$$d^* = \sum_{r=1}^{R} d_r^*$$

be the corresponding aggregated allocation. Furthermore, let

$$x_r^* = \sum_{i=1}^{N} d_r^*(i) P_{tr}(i, d^*)$$

be the average throughput of user r, $r = 1, ..., R$ under $(d_1^*, ..., d_R^*)$, and define

$$u_{N+1} = \max_{r=1,...,R} u_{r,\max}.$$

Then the following holds. When $(d_1^*, ..., d_R^*)$,$d_r \in \Re_+^N$, $r = 1, ..., R$, is an equilibrium allocation, then there exists a priority $i_0 \in \{1, ..., N\}$ such that, for $r = 1, ..., R$, we have

(a) $d_r^*(i) = 0$, for all $i \notin \{i_0, i_0 + 1\}$;

(b) $P_{tr}(i_0, d^*) \geq \frac{u_{i_0}}{u_{i_0+1}}$;

(c) $x_r^* = D_r(u^*)$ where

$$u^* = \frac{u_{i_0}}{P_{tr}(i_0, d^*)}.$$

(d) $\sum_{r=1}^{R} x_r^* = C$.

A proof of Proposition 1 is given in Appendix A. Using Proposition 1, we can characterize an equilibrium allocation as follows.

(a) states that, under every equilibrium allocation, at most two priorities are used. This result has the following interpretation. Let $(d_1^*, ..., d_R^*)$ be a given allocation vector, let $d^* = \sum_{r=1}^{R} d_r^*$ be the corresponding aggregated demand, and let the priority i^* be such that

$$P_t(i, d^*) = 0, \text{ for all } i < i^* \quad \text{and} \quad P_t(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Under the link model given in Section 2, packets with a priority lower than priority i^* will never be transmitted. As users are charged for every submitted packet, it is not economical to use these priorities. Furthermore, packets with a priority higher than $i^* + 1$ will be be transmitted with the same probability as packets with priority $i^* + 1$, but at a higher cost; and these priorities won't be used either.

(b) states that when priority i is used under a given equilibrium allocation $(d_1^*, ..., d_R^*)$, then the probability that a submitted packet with priority i gets transmitted is lower-bounded by u_i/u_{i+1}. This result has the following interpretation. In equilibrium, the average price for transmitting a packet in priority i is equal to $u_i/P_{tr}(i, d^*)$. When

$$P_{tr}(i, d*) > \frac{u_i}{u_{i+1}},$$

then it is more economical for users to switch their packets from priority i to priority $i + 1$. Therefore, Property (b) will hold in equilibrium.

(c) together with (d) states that there exits a unique price $u^* \in \Re_+$ such that for every user $r = 1, ..., R$, the average throughput x_r^* is given by

$$x_r^* = D_r(u^*).$$

To see this, note that by Assumption 1 the demand $D(u)$ is strictly decreasing on $[0, u_{N+1}]$.

(d) states that under every equilibrium allocation the whole link capacity is used.

In the next proposition, we derive sufficient conditions for an equilibrium allocation, which we then use to show that there always exists an equilibrium allocation.

Proposition 2 *Let Assumption 1 and 2 hold, and let*

$$u_{r,\max} = G_r(0), \qquad r = 1, ..., R.$$

Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let

$$d^* = \sum_{r=1}^{R} d_r^*$$

be the corresponding aggregated allocation. Furthermore, let

$$x_r^* = \sum_{i=1}^{N} d_r^*(i) P_{tr}(i, d^*)$$

be the average throughput of user r, $r = 1, ..., R$ under $(d_1^*, ..., d_R^*)$. Define

$$u_{N+1} = \max_{r=1,...,R} u_{r,\max},$$

and let i^* be such that

$$P_t(i, d^*) = 0, \text{ for all } i < i^* \quad \text{and} \quad P_t(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Then $(d_1^*, ..., d_R^*)$ is an equilibrium allocation when the following properties hold for $r = 1, ..., R$.

(a) $d_r^*(i) = 0$, *for* $i \neq i^*$.

(b) $d_r^*(i^*) = \frac{1}{P_{tr}(i^*,d^*)} D_r\left(\frac{u_{i^*}}{P_{tr}(i^*,d^*)}\right)$.

(c) $P_{tr}(i^*,d^*) \geq \frac{u_{i^*}}{u_{i^*+1}}$.

(d) $G_r(x_r^*)P_t(i^*,d^*) - u_{i^*} \leq 0$, *where we have equality when* $d_r(i^*) > 0$.

A proof for Proposition 2 is given in Appendix B. Furthermore, we show in Appendix B what we can use Proposition 2 to construct an equilibrium allocation and to obtain the following result.

Proposition 3 *Let Assumption 1 and 2 hold. Then there exists an equilibrium allocation. Furthermore, when there exists a priority* $i_0 \in \mathcal{I}$ *such that*

$$D(u_{i_0}) > C > D(u_{i_0+1}),$$

then the equilibrium is unique.

Proposition 3 states that we have a unique equilibrium allocation when there exists a priority $i_0 \in \mathcal{I}$ such that

$$D(u_{i_0}) = \sum_{r=1}^{R} D_r(u_{i_0}) > C > \sum_{r=1}^{R} D_r(u_{i_0+1}) = D(u_{i_0}).$$

When there exists a priority i_0 such that

$$D_r(u_{i_0}) = C,$$

then it can been shown that there is an infinite number of equilibrium allocations (see Appendix B).

3.2 Discussion of Propositions 1

Property (b) of Proposition 1 implies that we can use pricing to lower-bound the transmission probabilities in each priority. This result suggests that pricing can be used for congestion control through a similar mechanism as in the congestion avoidence algorithm TCP. Under both schemes, users receive a (congestion) feedback which indicates whether their packets have been transmitted or not. However, while TCP assumes that users will reduce their rate volunterialy when packet loss is detected, the priority service that we study here uses pricing to ensure that users indeed react appropriately to congestion. In this sense, this priority service provides a congestion avoidance mechanism that can be implemented in a non-cooperative environment where users behave in a selfish manner, and are only interested in maximizing their own net benefit.

Property (c) and (d) of Proposition 1, imply that there exists a unique price u^* such that for every user $r = 1, ..., R$, the average throughput is given by

$$x_r^* = D_r(u^*).$$

It is interesting to observe that the price u^* is equal to the congestion price users would pay under the smart market proposal by MacKie-Mason and Varian [10, 11]. Therefore, despite the differences between the smart market and the priority service that we consider here (see Section 1), users pay (in equilibrium) the same (congestion) price per (transmitted) packet, and obtain the same average throughput.

4 QoS for Packet Loss

In this section, we extend our analysis and consider users who perceive quality of service as function of the average r throughput and the packet loss rate. We model this situation by adding to the user's utility function a (penalty) term which depends on the packet transmission probabilities $P_{tr}(i, \cdot)$.

Consider a fixed customer r with a utility function U_r for which Assumption 1 holds. In addition, assume that $d_r = (d_r(1), ..., d_r(N)$ is the allocation of customer r and $d = (d(1), ..., d(N))$ is the aggregate allocation over all users. We then associated with user r the total utility

$$U_r(x_r) - \sum_{i=1}^{N} c_r\Big(P_{tr}(i,d)\Big)d_r(i), \tag{1}$$

where

$$x_r = \sum_{r=1}^{R} P_{tr}(i,d)d_r(i)$$

and the function $c_r : [0,\infty) \to [0,\infty)$ satisfies the following assumption.

Assumption 3 *The function $c_r : [0,\infty) \to [0,\infty)$ is bounded, non-increasing, and continuous, with*

$$c_r(\rho) = 0, \qquad \textit{for } \rho \geq 1.$$

The function c_r models the QoS requirements of user r with respect to packet loss. When c_r is equal to 0 for all $\rho \geq 0$, then we obtain the utility function of the previous section where users only cared about throughput. A user with with tight QoS requirements for packet loss can be modeled by letting the function $c_r(\rho)$ increase sharply when ρ falls below 1.

For this case, we wish to identify an allocation $(d_1^*, ..., d_R^*)$ such that for every user $r = 1, ..., R$, d_r^* is an optimal solution to the optimization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^{N} c_r\Big(P_{tr}(i,d)\Big)d_r(i) - \sum_{i=1}^{N} d_r(i)u_i \right\},$$

where

$$x_r = \sum_{i=1}^{N} d_r(i)P_{tr}(i,d^*).$$

We have the following results.

Proposition 4 *Let Assumption 1, 2, and 3 hold. In particular, let*

$$u_{r,\max} = G_r(0), \qquad r = 1, ..., R.$$

Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let

$$d^* = \sum_{r=1}^{R} d_r^*$$

be the corresponding aggregated allocation. Furthermore, let

$$x_r^* = \sum_{i=1}^{N} d_r^*(i)P_{tr}(i,d^*)$$

be the average throughput of user r, $r = 1, ..., R$ under $(d_1^*, ..., d_R^*)$, and define

$$u_{N+1} = \max_{r=1,\dots,R} u_{r,\max}.$$

Then the following holds. When $(d_1^*, ..., d_R^*)$,$d_r \in \Re_+^N$, $r = 1, ..., R$, is an equilibrium allocation, then there exists a priority $i_0 \in \{1, ..., N\}$ such that, for $r = 1, ..., R$, we have *(a)* $d_r^*(i) = 0$*, for all* $i \notin \{i_0, i_0+1\}$*;*

(b) $P_{tr}(i_0, d^*) \geq \frac{u_{i_0}}{u_{i_0+1}}$*;*

(c) $x_r^* = D_r(u_r^*)$*, for*

$$u_r^* = \min\left\{ u_{i_0+1}, \frac{u_{i_0} + c_r\Big(P_{tr}(i_0, d^*)\Big)}{P_{tr}(i_0, d^*)} \right\};$$

Proposition 4 is the equivalent of Proposition 1, however the congestion price u_r^* now depends on the QoS requirements of user r through the penalty term $c_r\Big(P_{tr}(i_0, d)\Big)$. This result has the following interpretation. Users with tight QoS requirement are willing to pay a higher price to obtain a better QoS service and submit their packets with priority $i_0 + 1$. Users with elastic traffic can tolerate packet loss and will submit their packets with priority i_0.

Proposition 5 *Let Assumption 1, 2, and 3 hold. Then there exists an equilibrium allocation.*

Compared with Proposition 3, we were not able to find (simple) conditions for the existence of a unique equilibrium.

5 Conclusions

The goal of this paper was to analyze the role of pricing in Differentiated Services (DiffServ) networks. For our study, we modeled DiffServ as a priority service, and assumed that users are free to choose the priorities they use and are charged accordingly. Too keep the analysis simple, we focused on the single link case, and used a bufferless, discrete-time link model. To capture the interaction among users, we modeled this situation as a non-cooperative game. For the case where only users with elastic traffic compete for the link capacity, we showed that there always exists an equilibrium allocation. Furthermore, this equilibrium allocation is unique, except for the (rare) case where there exists a priority $i \in \mathcal{I}$ such that the total demand $D(u_i)$ under the price u_i is equal to the link capacity. In addition, our analysis revealed a surprisingly simple relation between pricing decisions and the packet loss probabilities. In particular, when a packet with priority i is submitted under an equilibrium allocation $(d_1^*, ..., d_R^*)$, then the packet transmission probability $P_{tr}(i, d^*)$ is lower bounded by u_i/u_{i+1}, where $d^* = (d^*(1,), ..., d^*(N))$ is the corresponding aggregate allocation. This result suggest that in DiffServ networks, pricing is an effective tool to control congestion. For the case where users with QoS-sensitive traffic access the link, we showed that the above priority service provides differentiated QoS, i.e. users with with tight QoS requirements are willing to pay a higher price to use high priorities and obtain a better QoS.

We used a model as simple as possible for the analysis in this paper. Currently, we are extending this work to the network case, and consider more realistic link models. For additional future work, it would be interesting to explore the question whether there exists an algorithm for (dynamically) updating the the users' allocations which converges to an equilibrium allocation.

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A Proof of Proposition 1

To prove Proposition 1, we use the following lemma.

Lemma 1 *Let Assumption 1 hold and let*

$$u_{r,\max} = G_r(0), r = 1, ..., R.$$

Furthermore, let $d^* \in \Re_+^N$ be a given aggregate allocation and let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*, \quad \text{and} \quad P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Let $d_r^* \in \Re_+^N$ be the allocation of a given user $r \in \{1, ..., R\}$, and let $x_r^* = \sum_{i=1}^N P_{tr}(i, d^*) d_r^*(i)$ be the corresponding average throughput. Then $d_r^* \in \Re_+^N$ is an optimal solution to the maximization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^N d_r(i) u_i \right\},$$

where

$$x_r = \sum_{i=1}^{N} P_{tr}(i, d^*) d_r(i),$$

if and only if the following holds.

(1) For $i \notin \{i^*, i^*+1\}$, we have $d_r^*(i) = 0$

(2) When $d_r^*(i^*) > 0$, then $G_r(x_r^*) = \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ and $P_{tr}(i^*, d^*) \geq \frac{u_{i^*}}{u_{i^*+1}}$.

(3) When $d_r^*(i^*+1) > 0$, then $G_r(x_r^*) = u_{i^*+1}$ and $P_{tr}(i^*, d^*) \leq \frac{u_{i^*}}{u_{i^*+1}}$.

(4) When $d_r(i^*) = d_r(i^*+1) = 0$, then $u_{r,\max} \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ and $u_{r,\max} \leq u_{i^*+1}$.

Furthermore, when $d_r^* \in \Re_+^N$ is an optimal solution to the above maximization problem, then we have that

(5) When $P_{tr}(i^*, d^*) > \frac{u_{i^*}}{u_{i^*+1}}$, then $d_r(i^*+1) = 0$.

Proof: We will show that Properties (1)-(4) are necessary and sufficient conditions for d_r^* to be an optimal solution for the maximization problem given in the proposition.

We use the following notation. Let $d_r^* \in \Re_+^N$ be as given in the statement of the lemma, and for $d_r \in \Re_+^N$ let the function $F_r(d_r, d^*)$ be given by

$$F_r(d_r, d^*) = U_r(x_r) - \sum_{i=1}^{N} d_r(i) u_i,$$

where

$$x_r = \sum_{i=1}^{N} P_{tr}(i, d^*) d_r(i).$$

Furthermore, for $r = 1, ..., R$, let $f_{r,i}(d_r, d^*)$ be the derivative of $F_r(d_r, d^*)$ with respect to $d_r(i)$, i.e. we have

$$f_{r,i}(d_r, d^*) = \lim_{\alpha \to 0} \frac{F_r(d_r + \alpha e_i, d^*) - F_r(d_r, d^*)}{\alpha}, \qquad i = 1, ..., N,$$

where e_i is the ith unit vector. These derivatives exist as, by Assumption 1, $U_r(x_r)$ is differentiable on $[0, \infty)$. In particular, we have

$$f_{r,i}(d_r, d^*) = U_r'(x_r) P_{tr}(i, d^*) - u_i,$$

where U_r' is the derivative of U_r (with respect to x_r). The first order conditions for $d_r \in \Re_+^N$ to be an optimal solution for the maximization problem given in the statement of the lemma are then given as follows,

$$f_{r,i}(d_r, d^*) \leq 0, \qquad \text{for } i = 1, ..., N, \tag{2}$$

where this equation holds with equality when $d_r(i) > 0$. In the following, we show these first order conditions imply the Properties (1)-(4) of Lemma 1, and that the first order conditions are sufficient conditions.

Let the aggregate allocation $d^* \in \Re_+^N$ and the priority i^* be as given in the statement of the Lemma, i.e. we have that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*, \quad \text{and} \quad P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Note that, for $i < i^*$ we have

$$f_{r,i}(d_r, d^*) = -u_i < 0, \qquad d_r \in \Re_+^N.$$

Furthermore, for $i < i^* + 1$ we have

$$f_{r,i}(d_r, d^*) = G_r(x_r) - u_i < G_r(x_r) - u_{i^*+1} = f_{r,i^*+1}(d_r, d^*), \qquad d_r \in \Re_+^N.$$

This implies the following. When

$$f_{r,i}(d_r, d^*) \leq 0, \qquad \text{for } i = i^*, i^* + 1,$$

then we have that

$$f_{r,i}(d_r, d^*) < 0, \qquad \text{for } i \notin \{i^*, i^* + 1\}.$$

Combining this result with Equation (2), we obtain that when $d_r^* \in \Re_+^N$ is an optimal solution, then we have that

$$d_r^*(i) = 0, \qquad \text{for } i \notin \{i^*, i^* + 1\}.$$

This establishes Property (1).

To satisfy Equation (2) for $i = i^*$ and $i = i^* + 1$, we need the following. When $d_r(i^*) > 0$, then we have that

$$f_{r,i^*}(d_r^*, d^*) = G_r(x_r) P_{tr}(i^*, d^*) - u_{i^*} = 0$$

and

$$f_{r,i^*+1}(d_r, d^*) = G_r(x_r) - u_{i^*+1} \leq 0.$$

These two equations imply that

$$G_r(x_r) = \frac{u_{i^*}}{P_{tr}(i^*, d^*)} \leq u_{i^*+1}$$

and

$$P_{tr}(i^*, d^*) \geq \frac{u_{i^*}}{u_{i^*+1}}.$$

This gives Property (2). Similarly, when $d_r(i^* + 1) > 0$, then we have that

$$G_r(x_r) = u_{i^*+1} \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$$

and

$$P_{tr}(i^*, d^*) \leq \frac{u_{i^*}}{u_{i^*+1}},$$

which gives Property (3).

Finally, when $d_r^* \in \Re_+^N$ is an optimal solution with $d_r(i^*) = d_r(i^* + 1) = 0$, then Property (1) implies that $x_r = 0$. Therefore, we need the following to satisfy Equation (2),

$$f_{r,i^*}(d_r, d^*) = G_r(0) P_{tr}(i^*, d^*) - u_{i^*} \leq 0$$

and

$$f_{r,i^*+1}(d_r, d^*) = G_r(0) - u_{i^*+1} \leq 0.$$

Combining these two equations, we obtain that $u_{r,\max} = G_r(0) \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ and $u_{r,\max} \leq u_{i^*+1}$. This gives Property (4).

So far, we have shown that when the first order conditions given by Equation (2) are satisfied, then the Properties (1)-(4) hold. Similarly, one can show that when the Properties (1)-(4) hold, then the first order conditions are satisfied. We omit a detailed derivation here.

Next, we show that the first order conditions are sufficient conditions for an allocation $d_r \in \Re_+^N$ to be an optimal solution. By Assumption 1, we have that the derivative of $G_r(x_r)$ is negative on $[0, \infty)$. It follows that for $d_r \in \Re_+^N$ we have

$$\begin{aligned}
\frac{\partial f_{r,i^*}(d_r, d^*)}{\partial d_r(i^*)} &= G_r(x_r)' P_{tr}(i^*, d^*)^2 < 0 \\
\frac{\partial f_{r,i^*}(d_r, d^*)}{\partial d_r(i^* + 1)} &= G_r(x_r)' P_{tr}(i^*, d^*) < 0 \\
\frac{\partial f_{r,i^*+1}(d_r, d^*)}{\partial d_r(i^*)} &= G_r(x_r)' P_{tr}(i^*, d^*) < 0 \\
\frac{\partial f_{r,i^*+1}(d_r, d^*)}{\partial d_r(i^* + 1)} &= G_r(x_r)' < 0.
\end{aligned}$$

This implies that the second order conditions with respect to $d_r(i^*)$ and $d_r(i^*+1)$ are satisfied. Also, remember that when the first order conditions hold, then we have that

$$f_{r,i}(d_r, d^*) < 0, \qquad i \notin \{i^*, i^*+1\}.$$

Combining these two results, it follows that the first order conditions given by the Properties (1)-(4) are indeed sufficient for d_r^* to be an optimal solution.

To show that Property (5) holds, we proceed as follows. Assume that $P_{tr}(i^*, d^*) > \frac{u_{i^*}}{u_{i^*+1}}$. Then we have that

$$f_{r,i^*}(d_r, d^*) = G_r(x_r)P_{tr}(i^*, d^*) - u_{i^*} > G_r(x_r) - u_{i^*+1} = f_{r,i^*+1}(d_r, d^*),$$

and the first order conditions given by Equation (2) imply that $d_r(i^*+1) = 0$. ■

Proof of Proposition 1:

Property (a): This property follows from Property (1) of Lemma 1.

Property (b): Note that when $P_{tr}(i^*, d^*) < 1$, then there exists a user $r \in \{1, ..., R\}$ such that $d_r^*(i^*) > 0$. Property (b) then follows from Property (2) of Lemma 1.

Property (c): Consider a fixed user $r \in \{1, ..., R\}$. When $d_r^*(i^*) > 0$, then Property (c) follows from Property (2) of Lemma 1. When $d_r^*(i^*+1) > 0$, then by combining Property (3) of Lemma 1 with Property (b), we obtain that

$$P_{tr}(i^*, d^*) = \frac{u_{i^*}}{u_{i^*+1}}$$

and

$$G_r(x_r^*) = u_{i^*+1} = \frac{u_{i^*}}{P_{tr}(i^*, d^*)}.$$

This implies Property (c). Finally, when $d_r^*(i^*) = d_r^*(i^*+1) = 0$, then from Property (a) we have that

$$x_r^* = 0,$$

and from Property (4) of Lemma 1 we have that

$$u_{r,\max} \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}.$$

As $D_r(u) = 0$ for all $u \geq u_{r,\max}$, it follows that

$$x_r^* = D_r\left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right).$$

Property (d): Note that

$$\begin{aligned}
\sum_{r=1}^{R} x_r^* &= \sum_{r=1}^{R} P_{tr}(i^*, d^*)d_r(i^*) + \sum_{r=1}^{R} d_r^*(i^*+1) \\
&= P_{tr}(i^*, d^*)\sum_{r=1}^{R} d_r(i^*) + \sum_{r=1}^{R} d_r^*(i^*+1) \\
&= \frac{C - \sum_{r=1}^{R} d_r^*(i^*+1)}{\sum_{r=1}^{R} d_r^*(i^*)} \sum_{r=1}^{R} d_r^*(i^*) + \sum_{r=1}^{R} d_r^*(i^*+1) = C.
\end{aligned}$$

■

B Proof of Proposition 2 and 3

Proof of Proposition 2: When the Properties (a)-(d) hold, then Lemma 1 applies for each allocation d_r^*, $r = 1, ..., R,$; and $(d_1^*, ..., d_R^*)$ is an equilibrium allocation. In particular, Property (1) of Lemma 1 follows from Property (a) of Proposition 2. Furthermore, when $d_r^*(i^*) > 0$, then by Property (b) of Proposition 2 we have that

$$x_r^* = P_{tr}(i^*, d^*) d_r^*(i^*, d^*) = D_r\left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right).$$

It then follows that

$$G_r(x_r^*)P_{tr}(i^*, d^*) - u_{i^*} = \left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right) P_{tr}(i^*, d^*) - u_{i^*} = 0,$$

and Property (2) of Lemma 1 holds for user r. When $d_r^*(i^*) = 0$, then by Property (b) of Proposition 2 we have that $u_{r,\max} \le \frac{u_{i^*}}{P_{tr}(i^*,d^*)}$. It follows that

$$G_r(x_r^*)P_{tr}(i^*, d^*) - u_{i^*} = G_r(0)P_{tr}(i^*, d^*) - u_{i^*} = \; u_{r,\max}P_{tr}(i^*, d^*) - u_{i^*} \le 0,$$

and Property (4) of Lemma 1 holds for user r. ■

Proof of Proposition 3: Using Proposition 2, we can construct an equilibrium allocation as follows. Let the priority i_0 be such that

$$D(u_{i_0}) > C, \text{ and } D(u_{i_0+1}) \le C.$$

Such a priority i_0 exists as, by Assumption 1 and 2, we have that

$$D(u_1) > C \text{ and } D(u_{N+1}) = 0 < C.$$

For $\rho \in (0, 1]$, consider the allocations $d_r(\rho)$, $r = 1, ..., R$, given by

$$d_r(i, \rho) = \begin{cases} \frac{1}{\rho} D_r\left(\frac{u_{i_0}}{\rho}\right), & \text{for } i = i_0 \\ 0, & \text{otherwise} \end{cases}$$

For $\rho = 1$, we obtain that

$$\frac{C}{\sum_{r=1}^R d_r(i_0, 1)} = \frac{C}{D(u_{i_0})} < 1.$$

Similarly, for $\rho = \frac{u_{i_0}}{u_{i_0+1}}$, we have that

$$\frac{C}{\sum_{r=1}^R d_r(i_0, \rho)} = \frac{C}{D(u_{i_0+1})} \frac{u_{i_0}}{u_{i_0+1}} \ge \frac{u_{i_0}}{u_{i_0+1}}.$$

Note that $d_r(\rho)$ is continuous on $(0, 1]$, and it follows that there exists a $\rho^* \in [u_{i_0}/u_{i_0+1}, 1]$ such that

$$\frac{C}{\sum_{r=1}^R d_r(i_0, \rho^*)} = \rho^*, \quad r = 1, ..., R.$$

By construction, the allocation vector $(d_1(\rho^*)), ..., d_R(\rho^*))$ satisfies all properties of Proposition 2. This establishes that there exists an equilibrium allocation.

Next, we assume that there exists a priority $i_0 \in \{1, ..., N\}$ be such that

$$D(u_{i_0+1}) < C < D(u_{i_0}),$$

and show that there exists a unique equilibrium. Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let

$$d^* = \sum_{r=1}^R d_r^*.$$

Furthermore, let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_t(i, d^*) = 0, \text{ for all } i < i^*, \quad \text{and} \quad P_t(i, d^*) > 0, \text{ for all } i \geq i^*.$$

When $i^* > i_0$, then $(d_1^*, ..., d_R^*)$ is not an equilibrium allocation. To see this, note that in this case

$$D(u) < C, \qquad \text{for all } u \in [u_{i^*}, u_{i^*+1}],$$

and either Property (b), or Property (d), of Proposition 1 will not hold. Similarly, when $i^* < i_0$, then $(d_1^*, ..., d_R^*)$ is not an equilibrium allocation as

$$D(u) > C, \qquad \text{for all } u \in [u_{i^*}, u_{i^*+1}].$$

Therefore, it suffices to consider the case where $i^* = i_0$.

For this case, assume that $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, and let u^* be such that

$$D(u^*) = C.$$

Note that by assumption we have that $D(u_{i_0}) > C > D(u_{i_0+1})$, and it follows that

$$u_{i^*} < u^* < u_{i^*+1}.$$

Properties (b)-(d) of Proposition 1, together with Property (5) of Lemma 1, then imply that

$$P_{tr}(i^*, d^*) > \frac{u_{i^*}}{u_{i^*+1}}$$

and

$$d_r^*(i^* + 1) = 0, \qquad \text{for } r = 1, ..., R.$$

Combining the above equation with Property (a) of Proposition 1, it follows that when $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, then we have, for r=1,...,R,

$$d_r^*(i) = 0, \qquad \text{when } i \neq i^*. \tag{3}$$

Combining Equation 3 with Property (b) and (c) of Proposition 1, we obtain that

$$x_r^* = \sum_{i=1}^{N} P_{tr}(i, d^*) d_r^*(i) = P_{tr}(i^*, d^*) d_r^*(i^*) = D_r(u^*),$$

and

$$\frac{u_{i^*}}{P_{tr}(i^*, d^*)} = u^*.$$

This implies that

$$d_r^*(i^*) = \frac{u^*}{u_{i^*}} D_r(u^*), \qquad \text{for } r = 1, ..., R.$$

Furthermore, from Assumption 1 (which implies that the total demand $D(u)$ is strictly decreasing on $[0, u_{\max}]$) it follows that the price u^* is uniquely given by the capacity C and the function $D(u)$; and we have that the allocation vector $(d_1^*, ..., d_R^*)$ is unique, i.e. we have that for each user $r = 1, ..., R$ the allocation d_r^* is uniquely given by

$$d_r^*(i^*) = \frac{u^*}{u_{i^*}} D_r(u^*).$$

This establishes that existence of a unique equilibrium allocation for the case where there exists a priority $i_0 \in \{1, ..., N\}$ be such that

$$D(u_{i_0+1}) < C < D(u_{i_0}).$$

■

In the above proof we have shown that when there exists a i_0 such that

$$D(u_{i_0}) > C > D(u_{i_0+1}),$$

then there exists a unique equilibrium. In Example 1, we show that when for a priority $i_0 \in \{1, ..., N\}$ we have that

$$D_r(u_{i_0}) = C,$$

then there exists an infinite number of equilibrium allocations.

Example 1 *Let the set $\mathcal{A}$ be given as follows,*

$$\mathcal{A} = \{(a_1, ..., a_R) \in \Re_+^N \mid a_r \leq D_r(u_{i_0}), r = 1, ..., R\}.$$

Using $\mathcal{A}$ we can construct an equilibrium allocation as follows. For $(a_1, ..., a_R) \in \mathcal{A}$, consider the following allocation vector. For each user $r = 1, ..., R$, let

$$d_r^*(i) = \begin{cases} a_r, & \text{for } i = i_0 \\ \frac{u_{i_0}}{u_{i_0-1}}\Big(D_r(u_{i_0}) - a_r\Big), & \text{for } i = i_0 - 1 \\ 0, \ \text{otherwise} & \end{cases}$$

To see that $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, note the following. Let

$$d^* = \sum_{r=1}^{R} d_r^*.$$

When $d^*(i_0 - 1) > 0$, then we have

$$\begin{aligned} P_{tr}(i_0 - 1, d^*) &= \frac{C - \sum_{r=1}^{R} d_r^*(i_0)}{\sum_{r=1}^{R} \frac{u_{i_0}}{u_{i_0-1}} d_r^*(i_0 - 1)} \\ &= \frac{u_{i_0-1}}{u_{i_0}} \cdot \frac{C - \sum_{r=1}^{R} d_r^*(i_0)}{\sum_{r=1}^{R} \Big(D_r(u_{i_0}) - a_r\Big)} \\ &= \frac{u_{i_0-1}}{u_{i_0}} \end{aligned}$$

and for $r = 1, ..., R$,

$$\begin{aligned} x_r^* &= d^*(i_0) + P_{tr}(i_0 - 1, d^*) d^*(i_0 - 1) \\ &= a_r + \frac{u_{i_0-1}}{u_{i_0}} \left[\frac{u_{i_0}}{u_{i_0-1}}\Big(D_r(u_{i_0}) - a_r\Big)\right] \\ &= D_r(u_{i_0}). \end{aligned}$$

This implies that the Properties (a)-(d) of Proposition 2 hold for each user $r = 1, ..., R$. Similarly, when $d^*(i_0 - 1) = 0$, then the Properties (a)-(d) of Proposition 2 hold for each user $r = 1, ..., R$.

C Proof of Proposition 4 and 5

To prove Proposition 4, we use the following lemma.

Lemma 2 *Let Assumption 1, 2, and 3 hold, and let*

$$u_{r,\max} = G_r(0), \quad r = 1, ..., R.$$

Furthermore, let $d^* \in \Re_+^N$ be a given aggregate allocation and let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*, \quad \text{and} \quad P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Let $d_r^* \in \Re_+^N$ be the allocation of a given user $r \in \{1, ..., R\}$, and let $x_r^* = \sum_{i=1}^N P_{tr}(i, d^*) d_r^*(i)$ be the corresponding average throughput. Then $d_r^* \in \Re_+^N$ is an optimal solution to the maximization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^N d_r(i) u_i - \sum_{i=1}^N d_r(i) c_r \left(P_{tr}(i, d^*) \right) \right\},$$

where

$$x_r = \sum_{i=1}^N P_{tr}(i, d^*) d_r(i),$$

if and only if the following holds.

(1) $d_r(i) = 0$, *for* $i \notin \{i^*, i^* + 1\}$.

(2) When $d_r^*(i^*) > 0$, *then* $G_r(x_r^*) = \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{P_{tr}(i^*, d^*)}$ *and* $P_{tr}(i^*, d^*) \geq \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$.

(3) When $d_r^*(i^* + 1) > 0$, *then* $G_r(x_r^*) = u_{i^*+1}$ *and* $P_{tr}(i^*, d^*) \leq \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$.

(4) When $d_r(i^*) = d_r(i^* + 1) = 0$, *then* $u_{\max,r} \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ *and* $u_{\max,r} \leq u_{i^*+1}$.

(5) When $P_{tr}(i^*, d^*) > \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$, *then* $d_r(i^* + 1) = 0$.

Proof: This lemma can be proven using the same argument as in Lemma 1 by and adapting the notation suitably. In particular, for $d_r \in \Re_+^N$, the function $F_r(d_r, d^*)$ is now defined as follows,

$$F_r(d_r, d^*) = U_r(x_r) - \sum_{i=1}^N d_r(i) u_i - \sum_{i=1}^N d_r(i) c_r \left(P_{tr}(i, d^*) \right),$$

where

$$x_r = \sum_{i=1}^N P_{tr}(i, d^*) d_r(i).$$

The derivatives $f_{r,i}(d_r, d^*)$. i=1,...,N, of $F_r(d_r, d^*)$ with respect to $d_r(i) \in \Re_+$, are then given by

$$f_{r,i}(d_r, d^*) = \begin{cases} -u_i - c_r(0), & \text{for } i < i^* \\ G_r(x_r) P_{tr}(i^*, d^*) - u_{i^*} - c_r(P_{tr}(i^*, d^*)) & \text{for } i = i^* \\ G_r(x_r) - u_i & \text{for } i > i^* \end{cases}$$

■

Lemma 3 *Let Assumption 1, 2, and 3 hold, and let*

$$u_{r,\max} = G_r(0), \qquad r = 1, ..., R.$$

Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let

$$d^* = \sum_{r=1}^R d_r^*$$

be the corresponding aggregated allocation. Furthermore, let

$$x_r^* = \sum_{i=1}^{N} d_r^*(i) P_{tr}(i, d^*)$$

be the average throughput of user r, $r = 1, ..., R$ under $(d_1^*, ..., d_R^*)$. Define

$$u_{N+1} = \max_{r=1,\ldots,R} u_{r,\max},$$

and let i^* be such that

$$P_t(i, d^*) = 0, \text{ for all } i < i^* \quad \text{and} \quad P_t(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Then $(d_1^*, ..., d_R^*)$ is an equilibrium allocation when the following properties hold for $r = 1, ..., R$.

(1) $d_r(i) = 0$, for $i \notin \{i^*, i^* + 1\}$.

(2) There exists a $\kappa = \in [0, 1]$ such that

$$d_r(i^*) = \begin{cases} \frac{1}{P_{tr}(i^*,d)} D_r\left(\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)}\right), & \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} < u_{i^*+1}, \\ \frac{\kappa}{P_{tr}(i^*,d)} D_r\left(u_{i^*+1}\right), & \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_r(i^*+1) = \begin{cases} D_r\left(u_{i^*+1}\right), & \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} > u_{i^*+1}, \\ (1-\kappa) D_r\left(u_{i^*+1}\right), & \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Note that when the Properties (1) and (2) hold, then Lemma 2 applies for each allocation d_r^*, $r = 1, ..., R$, and $(d_1^*, ..., d_R^*)$ is an equilibrium allocation. In particular, Property (1) of Lemma 2 holds as, by definition, we have that $d_r(i) = 0$, for $i \notin \{i^*, i^* + 1\}$. Furthermore, when $d_r^*(i^*) > 0$ and $\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} < u_{i^*+1}$, then we have that $d_r(i^* + 1) = 0$ and

$$x_r^* = P_{tr}(i^*) d_r(i^*) = D_r\left(\frac{u_{i^*} + c_r(P_{tr}(i^*, d))}{P_{tr}(i^*, d)}\right).$$

It follows that

$$G_r(x_r^*) P_{tr}(i^*, d^*) - u_{i^*} - c_r(P_{tr}(i^*, d)) = \left(\frac{u_{i^*} + c_r(P_{tr}(i^*, d))}{P_{tr}(i^*, d)}\right) P_{tr}(i^*, d^*) - u_{i^*} - c_r(P_{tr}(i^*, d)) = 0,$$

and Property (2) of Lemma 2 holds. Similarly, one can show that Property (2) of Lemma 2 when $d_r^*(i^*) > 0$ and $\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}$; and that Property (3) of Lemma 2 is satisfied when $d_r^*(i^* + 1) > 0$. Finally, Property (4) of Lemma 2 holds by the definition of the allocations d_r^*, $r = 1, ..., R$.

■

Proof of Proposition 4: Property (a)-(d) of Proposition 4 can be proven by the same argument as the one used to prove Proposition 1. In particular, Lemma 2 implies that

$$P_{tr}(i^*, d^*) \geq \min_{r \in \{1,\ldots,R\}} \frac{u_{i^*} - c_r(P_{tr}(i^*, d))}{u_{i^*+1}} \geq \frac{u_{i^*}}{u_{i^*+1}}.$$

■

Proof of Proposition 5: To prove the existence of a local equilibrium allocation, we can construct an equilibrium allocation using Lemma 3. We proceed similar as in the proof of Proposition 3.

Let the priority i_0 be such that

$$D(u_{i_0}) > C, \text{ and } D(u_{i_0+1}) \leq C.$$

Such a priority i_0 exists as by Assumption 1 and 2, we have that

$$D(u_1) > C \text{ and } D(u_{N+1}) = 0 < C.$$

For $\rho \in (0,1]$, let the allocations $d_r(\rho)$, $r = 1, ..., R$, be given by

$$d_r(i_0, \rho) = \begin{cases} \frac{1}{\rho} D_r\left(\frac{u_{i_0}+c_r(\rho)}{\rho}\right), & \text{if } \frac{u_{i_0}+c_r(\rho)}{\rho} \leq u_{i_0+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_r(i_0+1, \rho) = \begin{cases} D_r\left(u_{i_0+1}\right), & \text{if } \frac{u_{i_0}+c_r(\rho)}{\rho} > u_{i_0+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_r(i, \rho) = 0$ for $i \notin \{i_0, i_0+1\}$. Observe that $d_r(\rho)$ is right-continuous on $(0,1]$. Furthermore, $d_r(\rho)$ is continuous on $(0,1]$, except at the unique point ρ_r for which

$$\frac{u_{i_0} + c_r(\rho_r)}{\rho_r} = u_{i_0+1}.$$

Such a ρ_r exists as the function $\frac{u_{i_0}+c_r(\rho)}{\rho}$ is strictly decreasing and continuous on $(0,1]$, with

$$\lim_{\rho \uparrow 1} \frac{u_{i_0} + c_r(\rho)}{\rho} = u_{i_0} < u_{i_0+1}$$

and

$$\lim_{\rho \downarrow (u_{i_0}/u_{i_0+1})} \frac{u_{i_0} + c_r(\rho)}{\rho} \geq \lim_{\rho \downarrow (u_{i_0}/u_{i_0+1})} \frac{u_{i_0}}{\rho} = u_{i_0+1}.$$

Using this result, the function

$$P(\rho) = \frac{C - \sum_{r=1}^{R} d_r(i_0+1, \rho)}{\sum_{r=1}^{R} d_r(i_0, \rho)}$$

is continuous on $[u_{i_0}/u_{i_0+1}, 1)$ expect at points ρ in the set $\mathcal{A} = \{\rho_r : r = 1, ..., R\}$. Let L be the cardinality of $\mathcal{A}$. Note that $L < R$, when for $r, r' \in \{1, ..., R\}$, $r \neq r'$, we have that

$$\rho_r = \rho_{r'}.$$

We define the vector α which orders the elements of A as follows. Let $\alpha = (\alpha(1), ..., \alpha(L)) \in (0,1)^L$ be such that $\alpha(l) < \alpha(l')$ for $l < l'$, and $\rho \in \{\alpha(l) : l = 1, ...L\}$ if and only if $\rho \in \mathcal{A}$. Using the vector α, we define the intervals $\mathcal{I}(l)$ by

$$\mathcal{I}(l) = [\alpha(l), \alpha(l+1)), \qquad l = 1, ..., L,$$

where we use the convention that $\alpha(L+1) = 1$. Note that $P(\rho)$ is continuous on $\mathcal{I}(l)$ for every $l = 1, ..., L$. Furthermore, observe that

$$\lim_{\rho \downarrow \alpha(1)} P\left(\alpha\right) = \frac{C - D(u_{i_0+1}) + \sum_{r:\rho_r = \alpha(1)} D_r(u_{i_0+1})}{(1/\alpha(1)) \sum_{r:\rho_r=\alpha(1)} D_r(u_{i_0+1})} = \alpha(1)\left(\frac{C - D(u_{i_0+1})}{\sum_{r:\rho_r=\alpha(1)} D_r(u_{i_0+1})} + 1\right) \geq \alpha(1).$$

It follows that there exists at least one $l \in \{1, ..., L\}$ such that $P(\alpha(l)) \geq \alpha(l)$. Let l^* be the largest such l. When

$$\lim_{\rho \uparrow \alpha(l^*+1)} < \alpha(l^*+1),$$

then we have that there exists a $\rho^* \in \mathcal{I}(l^*)$ such that

$$P(\rho^*) = \rho^*.$$

Note that in this case Lemma 3 applies to the allocation vector $d(\rho^*) = (d_r(\rho^*), ..., d_r(\rho^*))$; and $d(\rho^*) = (d_r(\rho^*), ..., d_r(\rho^*))$ is an equilibrium allocation.

Therefore, to prove Proposition 5, it suffices to consider the case where

$$\lim_{\rho \uparrow \alpha(l^*+1)} P(\rho) \geq \alpha(l^* + 1).$$

Note that

$$P\Big(\alpha(L+1)\Big) = P(1) = \frac{C}{\sum_r D_r(u_{i_0})} < 1,$$

and it follows that

$$P(\alpha(l)) < \alpha(l) \qquad \text{for } l = l^* + 1, ..., L + 1.$$

In particular, we have that

$$P(\alpha(l^* + 1)) < \alpha(l * +1).$$

For $\rho^* = \alpha(l^* + 1)$ and $\kappa \in [0, 1]$, let the allocations $d_r(\rho^*, \kappa)$, $r = 1, ..., R$, be given by

$$d_r(i_0, \rho^*, \kappa) = \begin{cases} \frac{\kappa}{\rho^*} D_r\left(u_{i_0+1}\right), & \text{if } \rho_r = \rho^*, \\ d_r(i_0, \rho^*) & \text{otherwise,} \end{cases}$$

$$d_r(i_0 + 1, \rho^*, \kappa) = \begin{cases} (1-\kappa) D_r\left(u_{i_0+1}\right), & \text{if } \rho_r = \rho^*, \\ d_r(i_0 + 1, \rho^*) & \text{otherwise,} \end{cases}$$

and $d_r(i, \rho^*, \kappa) = 0$ for $i \notin \{i_0, i_0 + 1\}$. Note that the allocations $d_r(\rho^*, \kappa)$, $r = 1, ..., R$, and the function

$$P(\rho^*, \kappa) = \frac{C - \sum d_r(i_0 + 1, \rho^*, \kappa)}{\sum d_r(i_0, \rho^*, \kappa)},$$

are continuous in κ on $[0, 1]$. Furthermore, we have that

$$\lim_{\kappa \uparrow 1} P(\rho^*, \kappa) = \lim_{\rho \uparrow \alpha(l^*+1)} P(\rho)$$

and

$$\lim_{\kappa \downarrow 0} P(\rho^*, \kappa) = P(\alpha(l^* + 1)).$$

As we have that

$$\lim_{\rho \uparrow \alpha(l^*+1)} P(\rho) \geq \alpha(l^* + 1) \quad \text{and} \quad P(\alpha(l^* + 1)) < \alpha(l^* + 1),$$

it follows that there exists a $\kappa^* \in [0, 1]$ such that

$$P(\rho^*, \kappa^*) = \alpha(l^* + 1).$$

Note that Lemma 3 applies to the allocation vector $d(\rho^*, \kappa^*) = (d_r(\rho^*, \kappa^*), ..., d_r(\rho^*, \kappa^*))$, and $d(\rho^*, \kappa^*) = (d_r(\rho^*, \kappa^*), ..., d_r(\rho^*, \kappa^*))$ is therefore an equilibrium allocation. This completes the proof.

■